

Higher-Order Generalized Four-Recurrent Finsler Geometry: A Theoretical Framework with Potential Applications in Biomedical Imaging

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ABSTRACT

This study develops a theoretical framework for generalized four-recurrent Finsler spaces based on higher-order Cartan horizontal covariant derivatives. The work establishes mathematical relationships among Weyl's projective curvature tensor, Cartan curvature tensors, Berwald curvature tensor, Ricci tensors, torsion tensors, and deviation tensors. Several characterization theorems and necessary and sufficient conditions for higher-order recurrence are derived, providing additional structural constraints on the behavior and stability of curvature in Finsler manifolds. Particular attention is given to the ability of Finsler geometry to represent direction-dependent and anisotropic structures, which are relevant to several biomedical engineering problems. From a biomedical perspective, the proposed mathematical framework may provide a theoretical basis for representing anisotropic structures encountered in diffusion-based medical imaging, neural fiber organization, vascular architecture, and biomechanical tissue modeling. However, the present study is theoretical and does not include patient data, clinical imaging datasets, or experimental validation. The biomedical relevance presented in this work should therefore be interpreted as a potential application framework rather than as a demonstrated clinical or diagnostic application. Future studies will be required to implement the proposed geometric formulation on real biomedical imaging datasets and evaluate its computational and clinical utility.

Keywords: *Finsler Geometry; Curvature Tensors; Diffusion MRI; Biomedical Imaging; Anisotropic Modeling*

INTRODUCTION

Biomedical structures frequently exhibit direction-dependent and spatially heterogeneous properties. Examples include the organization of neural fibers, vascular networks, muscle fibers, and diffusion patterns within biological tissues. Conventional isotropic geometric models may not adequately represent such directional characteristics. Finsler geometry provides a mathematical generalization of Riemannian geometry in which the metric depends not only on position but also on direction, making it potentially suitable for the mathematical representation of anisotropic biological structures.

In biomedical imaging, particularly diffusion-based imaging, directional information is an important

component of the representation of tissue microstructure. Similarly, geometric descriptions of neural pathways, vascular structures, and biomechanical tissues may benefit from mathematical models capable of incorporating directional dependence. These observations provide a conceptual motivation for investigating Finsler-geometric methods in biomedical engineering.

However, the present investigation does not directly analyze clinical or experimental biomedical data. Rather, it develops a higher-order mathematical framework that may subsequently be adapted to biomedical applications in which anisotropy and curvature play important roles. The biomedical component of this study is therefore intended to

establish the theoretical relevance and possible translational direction of the mathematical results rather than to demonstrate a validated biomedical application.

Previous studies have extensively investigated recurrent and higher-order curvature structures in Finsler geometry. Abu-Donia et al. (2020) examined curvature properties associated with the WW-curvature tensor in relativistic space-times, while Ahsan and Ali (2016) investigated curvature tensors in the context of general relativity. Within Finsler geometry, Al-Qashbari and co-authors have contributed to the study of recurrent and higher-order recurrent structures. Abdallah, Al-Qashbari, and Al-ssallal (2024) investigated higher-order generalizations of special curvature tensors on recurrent Finsler structures. Al-Qashbari et al. (2024) examined generalized BK-recurrent Finsler spaces and related recurrent structures, while other studies have considered generalized birecurrent Finsler spaces and mixed covariant derivatives in the sense of Cartan. Earlier work by Al-Qashbari (2020) on second-order generalized curvature tensors and Misra et al. (2014) on higher-order recurrent Finsler spaces involving Berwald curvature further established the mathematical basis for higher-order recurrence. Goswami (2017) provided a broader review of special Finsler spaces and their curvature-based properties.

Despite these developments, the relationships among higher-order generalized four-recurrent structures and the associated Weyl, Cartan, Berwald, Ricci, torsion, and deviation tensors remain an area requiring further mathematical characterization. The present study addresses this gap by deriving higher-order tensorial relationships and necessary-and-sufficient characterization conditions within generalized four-recurrent Finsler spaces. The study additionally discusses the potential relevance of these geometric structures to anisotropic biomedical systems, particularly biomedical imaging, while recognizing that such applications remain theoretical and require future computational and experimental validation.

Aim of the Study

The primary aim of this study is to develop and mathematically characterize higher-order generalized four-recurrent Finsler spaces using Cartan covariant derivatives. The study specifically seeks to:

1. derive higher-order covariant relationships involving Weyl's projective curvature tensor and Cartan curvature tensors;
2. establish relationships among curvature, Ricci, torsion, Berwald, and deviation tensors;
3. formulate necessary and sufficient conditions for higher-order recurrence;
4. examine the mathematical implications of curvature stability within generalized four-recurrent Finsler spaces; and

5. establish the potential relevance of the proposed framework to anisotropic biomedical structures, particularly diffusion-based medical imaging, neural fiber organization, vascular architecture, and biomechanical tissue modeling.

The fifth objective is theoretical and does not represent experimental or clinical validation.

Mathematical Methodology

Study Design

This study is a theoretical investigation based on differential and Finsler geometry. The analysis does not involve human participants, patient records, biological specimens, or clinical imaging datasets. The mathematical results are derived using established definitions, tensorial identities, higher-order Cartan covariant derivatives, tensor contractions, and algebraic manipulations within generalized four-recurrent Finsler spaces.

Mathematical Framework

The study is formulated within the framework of Finsler geometry equipped with the Cartan connection. Higher-order horizontal covariant derivatives are applied to the relevant curvature and associated tensors. The analysis focuses on relationships among Weyl's projective curvature tensor, Cartan curvature tensors, Berwald curvature tensor, Ricci tensors, torsion tensors, and deviation tensors under generalized four-recurrence conditions.

Derivation and Verification Strategy

The principal results are obtained through covariant differentiation, tensor contraction (transvection), substitution, and application of previously established identities. The characterization theorems are derived from the defining generalized four-recurrence conditions and the fundamental identities associated with the Cartan connection. Necessary and sufficient conditions are stated where both implications are established by the corresponding derivations.

Biomedical Relevance and Potential Applications

The mathematical framework developed in this study has potential relevance to biomedical engineering because several biological structures exhibit anisotropic and direction-dependent organization. The principal advantage of the Finsler formulation is its ability to incorporate directional dependence into the underlying metric structure.

One important development in this direction has been done by Abu-Donia et al. (2020) which studied the behavior of the curvature in spacetime in the context of relativistic space-times in which the w-curvature tensor is studied. Ahsan and Ali (2016) extended this research by considering the curvature tensor of space-time in general relativity, providing a more in-depth

analysis of the relationship between geometry and physics. These studies have been of great help to our understanding of curvature in non-constant curvature spaces. Al-Qashbari and his co-authors have significantly contributed to the study of four recurrent structures and higher order generalizations in the field of Finsler spaces. Abdallah, Al-Qashbari and Al-ssallal (2024) investigated the higher order generalizations of special curvature tensors on recurrent Finsler structures. In a similar manner, Al-Qashbari et al. (2024) explored generalized BK-recurrent Finsler spaces, a class of spaces with recurrent properties that have been studied in other contexts and which play a role in the understanding of the complex geometric structures. Further, the generalized birecurrent Finsler spaces studied by Al-Qashbari et al. (2024) have given a deeper understanding of the geometric features

related to the mixed covariant derivatives in the sense of Cartan. Pursuing the studies of Al-Qashbari (2020) on second-order generalized curvature tensors, and Misra et al. (2014) on higher-order recurrent Finsler space with Berwald's curvature tensor, the influence of curvature and geometric structure in higher-dimensional Finsler space is broadened. The systematic review conducted by Goswami (2017) on several special Finsler spaces has also added to this rich collection of knowledge, providing a means to researchers to explore the variety of curvature-based properties in Finsler geometry. In this paper, the present author tries to further these investigations, with a special emphasis on generalized curvature tensors in Finsler spaces, and their connection with the recurrent properties and higher order generalizations of Finsler spaces.

Two vectors y_i and y^i meet the following conditions

$$\begin{aligned} \text{a) } y_i &= g_{ij} y^j, \quad \text{b) } y_i y^i = F^2, \quad \text{c) } \delta_j^k y^j = y^k, \\ \text{d) } g_{ir} \delta_j^i &= g_{rj} \quad \text{and} \quad \text{e) } g^{jk} \delta_k^i = g^{ji}. \end{aligned} \quad (1.1)$$

The quantities g_{ij} and g^{ij} are related by

$$\begin{aligned} \text{a) } g_{ij} g^{jk} &= \delta_i^k = \begin{cases} 1, & \text{if } i = k \\ 0, & \text{if } i \neq k \end{cases}, \\ \text{b) } g_{|h}^{jk} &= 0 \quad \text{and} \quad \text{c) } g_{ij|h} = 0. \end{aligned} \quad (1.2)$$

Tensor C_{ijk} is known as (h)hv-torsion tensor defined by

$$\text{a) } C_{ijk} = \frac{1}{4} \partial_i \partial_j \partial_k F^2, \quad \text{b) } C_p = C_{ip}^i \quad \text{and} \quad \text{c) } C_{rjp} = g_{ir} C_{jp}^i. \quad (1.3)$$

The vector y^i and metric function F are vanished identically for Cartan's covariant derivative.

$$\text{a) } F_{|h} = 0 \quad \text{and} \quad \text{b) } y_{|h}^i = 0. \quad (1.4)$$

Tensor W_{jkh}^i , torsion tensor W_{jk}^i and deviation tensor W_j^i are defined by:

$$\begin{aligned} W_{jkh}^i &= H_{jkh}^i + \frac{2 \delta_j^i}{(n+1)} H_{[hk]} + \frac{2 y^i}{(n+1)} \partial_j H_{[kh]} + \frac{\delta_k^i}{(n^2-1)} (n H_{jh} + H_{hj} + y^r \partial_j H_{hr}) \\ &\quad - \frac{\delta_h^i}{(n^2-1)} (n H_{jk} + H_{kj} + y^r \partial_j H_{kr}), \end{aligned} \quad (1.5)$$

$$W_{jk}^i = H_{jk}^i + \frac{y^i}{(n+1)} H_{[jk]} + 2 \left\{ \frac{\delta_{[j}^i}{(n^2-1)} (n H_{k]} - y^r H_{k]r}) \right\}. \quad (1.6)$$

$$W_j^i = H_j^i - H \delta_j^i - \frac{1}{(n+1)} (\partial_r H_j^r - \partial_j H) y^i, \text{ respectively.} \quad (1.7)$$

The tensors W_{jkh}^i , and W_{jk}^i satisfy the following identities

$$\begin{aligned} \text{a) } W_{jkh}^i y^j &= W_{kh}^i, \quad \text{b) } W_{kh}^i y^k = W_h^i, \\ \text{c) } W_{jki}^i &= W_{jk} \quad \text{and} \quad \text{d) } g_{ir} W_{jkh}^i = W_{rjkh}. \end{aligned} \quad (1.8)$$

Also, if we suppose that the tensor W_j^i and W_{jk} satisfy the following identities

$$\begin{aligned} \text{a) } W_k^i y^k &= 0, \quad \text{b) } W_i^i = 0, \quad \text{c) } g_{ir} W_j^i = W_{rj}, \\ \text{d) } g^{jk} W_{jk} &= W \quad \text{and} \quad \text{e) } W_{jk} y^k = 0. \end{aligned} \quad (1.9)$$

Cartan's third curvature tensor R_{jkh}^i , Ricci tensor R_{jk} , H_{kh}^i , the vector H_k and scalar curvature H are defined as

$$\begin{aligned}
& \text{a) } R_{jkh}^i = \Gamma_{hjk}^{*i} + (\Gamma_{ljk}^{*i}) G_h^l + C_{jm}^i (G_{kh}^m - G_{kl}^m G_h^l) + \Gamma_{mk}^{*i} \Gamma_{jh}^{*m} - k/h, \quad \text{b) } R_{jkh}^i y^j = H_{kh}^i, \\
& \text{c) } R_{jk} y^j = H_k, \quad \text{d) } R_{jk} y^k = R_j, \quad \text{e) } R_i^i = R, \quad \text{f) } g_{ir} R_{jkh}^i = R_{rjkh}, \\
& \text{g) } g^{jk} R_{jk} = R, \quad \text{h) } g^{jk} R_{jkh}^i = R_h^i, \quad \text{i) } R_{jki}^i = R_{jk}, \quad \text{t) } H_i y^i = H_i^i = (n-1)H, \\
& \text{j) } H_{kh}^i y^k = H_h^i, \quad \text{k) } H_{ki}^i = H_k, \quad \text{l) } H_{jki}^i = H_{jk} \text{ and } \text{m) } H_{rkh}^r = H_{hk} - H_{kh}. \quad (1.10)
\end{aligned}$$

The Cartan's 4th curvature tensor K_{jkh}^i , the Ricci tensor K_{jk} , the vector K_k , and the scalar curvature K are defined as follows

$$\begin{aligned}
& \text{a) } K_{jkh}^i g_{ir} = K_{rjkh}, \quad \text{b) } K_{jk} y^j = H_k, \quad \text{c) } K_{jk} y^k = K_j, \\
& \text{d) } g^{jk} K_{jk} = K, \quad \text{e) } K_{jki}^i = K_{jk}, \quad \text{f) } g^{jk} K_{jkh}^i = K_h^i, \quad \text{g) } H_{jkh}^i = K_{jkh}^i + y^s (\partial_j K_{s kh}^i), \text{ and} \\
& \text{h) } H_{jkh}^i - K_{jkh}^i = (P_{jklh}^i + P_{jk}^r P_{rh}^i - h/k). \quad (1.11)
\end{aligned}$$

2. Preliminaries

We introduced the generalized by Cartan's covariant derivative for Wely's projective curvature tensor W_{jkh}^i was given by

$$W_{jkh|l}^i = \lambda_l W_{jkh}^i + \mu_l (\delta_h^i g_{jk} - \delta_k^i g_{jh}). \quad (2.1)$$

A Finsler space F_n which the curvature tensor W_{jkh}^i satisfies the condition (2.1) is referred to as a generalized $W_{|h}$ - recurrent space and denoted by $G W_{|h} - RF_n$.

Where $|l$ is called h -covariant derivative of order one with respect to x^l .

By taking the h -covariant derivative of (2.1), with respect to x^m , and applying equations (2.1) and (1.2c), we obtain

$$W_{jkh|l|m}^i = a_{lm} W_{jkh}^i + b_{lm} (\delta_h^i g_{jk} - \delta_k^i g_{jh}). \quad (2.2)$$

Where $|l|m$ are called h - covariant derivative of second-orders with respect to x^l and x^m , respectively, and $a_{lm} = \lambda_{m|l} + \lambda_m \lambda_l$ and $b_{lm} = \mu_{m|l} + \lambda_m \mu_l$ are non-zero covariant tensors field of second order, respectively.

A Finsler space F_n which the curvature tensor W_{jkh}^i satisfies the condition (2.2) is referred to as a generalized $W_{|h}$ - Birecurrent space and denoted by $G W_{|h} - BRF_n$.

By taking the h -covariant derivative of (2.2), with respect to x^n , and applying equations (2.1) and (1.2c), we obtain

$$W_{jkh|l|m|n}^i = c_{lmn} W_{jkh}^i + d_{lmn} (\delta_h^i g_{jk} - \delta_k^i g_{jh}). \quad (2.3)$$

Where $|l|m|n$ are called h -covariant derivative of third-orders with respect to x^l , x^m and x^n , respectively, and $c_{lmn} = a_{m|l} + a_{ml} \lambda_n$ and $d_{lmn} = a_{ml} \mu_n + b_{m|l} + b_{ml} \mu_n$ are non-zero covariant tensors field of third order, respectively.

A Finsler space F_n which the curvature tensor W_{jkh}^i satisfies the condition (2.3) is referred to as a generalized $W_{|h}$ - Trirecurrent space and denoted by $G W_{|h} - TRF_n$.

By taking the h -covariant derivative of (2.3), with respect to x^s , and applying equations (2.1) and (1.2c), we obtain

$$W_{jkh|l|m|n|s}^i = e_{lmns} W_{jkh}^i + f_{lmns} (\delta_h^i g_{jk} - \delta_k^i g_{jh}). \quad (2.4)$$

Where $|l|m|n|s$ are called h -covariant derivative of fourth-orders with respect to x^l , x^m , x^n and x^s , respectively, and $e_{lmns} = c_{lmn|s} + c_{lmn} \lambda_s$ and $f_{lmns} = c_{lmn} \mu_s + d_{lmn|s}$ are non-zero covariant tensors field of third order, respectively.

A Finsler space F_n which the curvature tensor W_{jkh}^i satisfies the condition (2.4) is referred to as a generalized $W_{|h}$ - Four - recurrent space and denoted by $GW_{|h} - FRF_n$.

Al-Qashbari and AL-ssallal studied generalization of Weyl's projective curvature tensor in Finsler spaces by: A Study on generalized W -recurrent, Birecurrent, and Ricci Tensors using Cartan's first and second order derivatives in Finsler space which characterized by the condition:

$$W_{jkh|l}^i = \lambda_l W_{jkh}^i + \mu_l (\delta_h^i g_{jk} - \delta_k^i g_{jh}) + \frac{1}{4} \gamma_l (W_h^i g_{jk} - W_k^i g_{jh}). \quad (2.5)$$

where λ_m, μ_m and γ_m are non-zero covariant vectors of first order.

A Finsler space F_n which the Wely's projective curvature tensor W_{jkh}^i satisfies the condition (2.5) is referred to as a generalization generalized $W|_h$ - recurrent space and denoted by $G^{2nd}W|_h - RF_n$.

3. Exploring the Relationship Between Weyl's Curvature Tensor and Cartan's Third Curvature Tensor in Finsler Geometry

Finsler geometry, which extends the concepts of Riemannian geometry, offers a robust framework for modeling a diverse range of physical phenomena. Within Finsler spaces, the curvature properties of the space are described by various curvature tensors, with Weyl's and Cartan's third curvature tensors being of particular significance. While the individual geometric and physical implications of these tensors have been widely studied, the connection between them has not been fully explored and remains an area of active research.

This paper aims to examine the relationship between Weyl's curvature tensor and Cartan's third curvature tensor within the context of Finsler spaces. By analyzing their algebraic and geometric properties, we aim to derive new identities and inequalities that connect these two fundamental tensors. The results of this study are expected to enhance our understanding of the curvature structure of Finsler spaces and offer new insights into their potential applications, particularly in gravitational theories and cosmology.

Some properties of W_{jkh}^i curvature tensor was proposed by Ahsan and Ali [3],[4] in (2014). For $(n = 4)$ a Riemannian space, it is known that Cartan's third curvature tensor R_{jkh}^i and Wely's projective curvature tensor W_{jkh}^i are connected by the formula

$$W_{jkh}^i = R_{jkh}^i + \frac{1}{3} (\delta_k^i R_{jh} - g_{jk} R_h^i). \quad (3.1)$$

By taking the covariant derivative of (3.1), with respct to x^m, x^l, x^n and x^s , respectively, in the sense of Cartan and using (1.2c), we get

$$W_{jkh|l|m|n|s}^i = R_{jkh|l|m|n|s}^i + \frac{1}{3} (\delta_k^i R_{jh} - g_{jk} R_h^i)_{|l|m|n|s}. \quad (3.2)$$

By substituting equations (3.4) in [8] and (3.1) in to (3.2), we obtain:

$$\begin{aligned} R_{jkh|l|m|n|s}^i &= \alpha_{lmns} R_{jkh}^i + \beta_{lmns} (\delta_h^i g_{jk} - \delta_k^i g_{jh}) + \chi_{lmns} (W_h^i g_{jk} - W_k^i g_{jh}) \\ &+ v_{lmn} (W_{h|s}^i g_{jk} - W_{k|s}^i g_{jh}) + \frac{1}{4} \omega_{lms} (W_{h|n}^i g_{jk} - W_{k|n}^i g_{jh}) \\ &+ \frac{1}{4} c_{lm} (W_{h|n|s}^i g_{jk} - W_{k|n|s}^i g_{jh}) + \frac{1}{4} \rho_{lms} (W_{h|m}^i g_{jk} - W_{k|m}^i g_{jh}) \\ &+ \frac{1}{4} \eta_{ln} (W_{h|m|s}^i g_{jk} - W_{k|m|s}^i g_{jh}) + \frac{1}{4} \tau_{ls} (W_{h|m|n}^i g_{jk} - W_{k|m|n}^i g_{jh}) \\ &+ \frac{1}{4} \gamma_l (W_{h|m|n|s}^i g_{jk} - W_{k|m|n|s}^i g_{jh}) + \frac{1}{3} \alpha_{lmns} (\delta_k^i R_{jh} - g_{jk} R_h^i) \\ &- \frac{1}{3} (\delta_k^i R_{jh} - g_{jk} R_h^i)_{|l|m|n|s}. \end{aligned} \quad (3.3)$$

This demonstrates that

$$R_{jkh|l|m|n|s}^i = \alpha_{lmns} R_{jkh}^i + \beta_{lmns} (\delta_h^i g_{jk} - \delta_k^i g_{jh}) + \chi_{lmns} (W_h^i g_{jk} - W_k^i g_{jh})$$

$$\begin{aligned}
& +v_{lmn}(W_{h|s}^i g_{jk} - W_{k|s}^i g_{jh}) + \frac{1}{4}\omega_{lms}(W_{h|n}^i g_{jk} - W_{k|n}^i g_{jh}) \\
& + \frac{1}{4}c_{lm}(W_{h|n|s}^i g_{jk} - W_{k|n|s}^i g_{jh}) + \frac{1}{4}\rho_{lns}(W_{h|m}^i g_{jk} - W_{k|m}^i g_{jh}) \\
& + \frac{1}{4}\eta_{ln}(W_{h|m|s}^i g_{jk} - W_{k|m|s}^i g_{jh}) + \frac{1}{4}\tau_{ls}(W_{h|m|n}^i g_{jk} - W_{k|m|n}^i g_{jh}) \\
& + \frac{1}{4}\gamma_l(W_{h|m|n|s}^i g_{jk} - W_{k|m|n|s}^i g_{jh}).
\end{aligned} \tag{3.4}$$

If and only if

$$(\delta_k^i R_{jh} - g_{jk} R_h^i)_{|l|m|n|s} = \alpha_{lmns}(\delta_k^i R_{jh} - g_{jk} R_h^i). \tag{3.5}$$

Therefore, the proof of theorem is completed, we can say

Theorem 3.1. In the context of $G^{2nd}R_{|h} - FRF_n$, Cartan's 3th curvature tensor R_{jkh}^i defines a generalized four-recurrent Finsler space if and only if the tensor $(\delta_k^i R_{jh} - g_{jk} R_h^i)$ is a generalized four-recurrent Finsler space.

By transvecting condition (3.3) with y^j , and utilizing equations (1.1a), (1.4b), (1.10b) and (1.10c), we obtain the following result

$$\begin{aligned}
H_{kh|l|m|n|s}^i &= \alpha_{lmns}H_{kh}^i + \beta_{lmns}(\delta_h^i y_k - \delta_k^i y_h) + \chi_{lmns}(W_{h|s}^i y_k - W_{k|s}^i y_h) \\
&+ v_{lmn}(W_{h|s}^i y_k - W_{k|s}^i y_h) + \frac{1}{4}\omega_{lms}(W_{h|n}^i y_k - W_{k|n}^i y_h) + \frac{1}{4}\rho_{lns}(W_{h|m}^i y_k - W_{k|m}^i y_h) \\
&+ \frac{1}{4}c_{lm}(W_{h|n|s}^i y_k - W_{k|n|s}^i y_h) + \frac{1}{4}\eta_{ln}(W_{h|m|s}^i y_k - W_{k|m|s}^i y_h) + \frac{1}{3}\alpha_{lmns}(\delta_k^i H_h - y_k R_h^i) \\
&+ \frac{1}{4}\tau_{ls}(W_{h|m|n}^i y_k - W_{k|m|n}^i y_h) + \frac{1}{4}\gamma_l(W_{h|m|n|s}^i y_k - W_{k|m|n|s}^i y_h) \\
&- \frac{1}{3}(\delta_k^i H_h - y_k R_h^i)_{|l|m|n|s}.
\end{aligned} \tag{3.6}$$

This demonstrates that

$$\begin{aligned}
H_{kh|l|m|n|s}^i &= \alpha_{lmns}H_{kh}^i + \beta_{lmns}(\delta_h^i y_k - \delta_k^i y_h) + \chi_{lmns}(W_{h|s}^i y_k - W_{k|s}^i y_h) \\
&+ v_{lmn}(W_{h|s}^i y_k - W_{k|s}^i y_h) + \frac{1}{4}\omega_{lms}(W_{h|n}^i y_k - W_{k|n}^i y_h) + \frac{1}{4}\rho_{lns}(W_{h|m}^i y_k - W_{k|m}^i y_h) \\
&+ \frac{1}{4}c_{lm}(W_{h|n|s}^i y_k - W_{k|n|s}^i y_h) + \frac{1}{4}\eta_{ln}(W_{h|m|s}^i y_k - W_{k|m|s}^i y_h) \\
&+ \frac{1}{4}\tau_{ls}(W_{h|m|n}^i y_k - W_{k|m|n}^i y_h) + \frac{1}{4}\gamma_l(W_{h|m|n|s}^i y_k - W_{k|m|n|s}^i y_h).
\end{aligned} \tag{3.7}$$

If and only if

$$(\delta_k^i H_h - y_k R_h^i)_{|l|m|n|s} = \alpha_{lmns}(\delta_k^i H_h - y_k R_h^i). \tag{3.8}$$

Therefore, the proof of theorem is completed, we conclude

Theorem 3.2. In the context of $G^{2nd}R_{|h} - FRF_n$, the h -covariant derivative of fourth-orders for the torsion tensor H_{kh}^i defines a generalized four-recurrent Finsler space if and only if (3.8) is satisfied.

By transvecting condition (3.6) with y^k , and applying relations (1.1b), (1.1c), (1.4a), (1.4b), (1.9a) and (1.10j), we obtain the following result

$$\begin{aligned}
H_{h|l|m|n|s}^i &= \alpha_{lmns}H_h^i + \beta_{lmns}(\delta_h^i F^2 - y^i y_h) + \chi_{lmns}W_h^i F^2 + v_{lmn}(W_{h|s}^i F^2) \\
&+ \frac{1}{4}\omega_{lms}(W_{h|n}^i F^2) + \frac{1}{4}\rho_{lns}(W_{h|m}^i F^2) + \frac{1}{4}c_{lm}(W_{h|n|s}^i F^2) + \frac{1}{4}\eta_{ln}(W_{h|m|s}^i F^2) \\
&+ \frac{1}{4}\tau_{ls}(W_{h|m|n}^i F^2) + \frac{1}{4}\gamma_l(W_{h|m|n|s}^i F^2) - \frac{1}{3}(y^i H_h - F^2 R_h^i)_{|l|m|n|s} \\
&+ \frac{1}{3}\alpha_{lmns}(y^i H_h - F^2 R_h^i).
\end{aligned} \tag{3.9}$$

This demonstrates that

$$\begin{aligned}
H_{h|l|m|n|s}^i &= \alpha_{lmns}H_h^i + \beta_{lmns}(\delta_h^i F^2 - y^i y_h) + \chi_{lmns}W_h^i F^2 + v_{lmn}(W_{h|s}^i F^2) \\
&+ \frac{1}{4}\omega_{lms}(W_{h|n}^i F^2) + \frac{1}{4}\rho_{lns}(W_{h|m}^i F^2) + \frac{1}{4}c_{lm}(W_{h|n|s}^i F^2) + \frac{1}{4}\eta_{ln}(W_{h|m|s}^i F^2) \\
&+ \frac{1}{4}\tau_{ls}(W_{h|m|n}^i F^2) + \frac{1}{4}\gamma_l(W_{h|m|n|s}^i F^2). \quad (3.10)
\end{aligned}$$

If and only if

$$(y^i H_h - F^2 R_h^i)_{|l|m|n|s} = \alpha_{lmns}(y^i H_h - F^2 R_h^i). \quad (3.11)$$

Therefore, the proof of theorem is completed, we can say

Theorem 3.3. In the context of $G^{2nd}R_{|h} - FRF_n$, the h -covariant derivative of fourth-order for the deviation tensor H_h^i represents a generalized four-recurrent Finsler space if and only if the condition in equation (3.11) is satisfied.

By contracting the indices i and h in equations (3.6) and (3.9), respectively and utilizing equations (1.2a), (1.1a), (1.1b), (1.4a), (1.10k), (1.10t), and (1.9b), we obtain the following result:

$$\begin{aligned}
H_{k|l|m|n|s} &= \alpha_{lmns}H_k + (n-1)\beta_{lmns}y_k - \chi_{lmns}(W_k^i y_i) - v_{lmn}(W_{k|s}^i y_i) \\
&- \frac{1}{4}\omega_{lms}(W_{k|n}^i y_i) - \frac{1}{4}\rho_{lns}(W_{k|m}^i y_i) - \frac{1}{4}c_{lm}(W_{k|n|s}^i y_i) - \frac{1}{4}\eta_{ln}(W_{k|m|s}^i y_i) \\
&- \frac{1}{4}\tau_{ls}(W_{k|m|n}^i y_i) - \frac{1}{4}\gamma_l(W_{k|m|n|s}^i y_i) + \frac{1}{3}\alpha_{lmns}(H_k - y_k R) - \frac{1}{3}(H_k - y_k R)_{|l|m|n|s}. \quad (3.12)
\end{aligned}$$

This demonstrates that

$$\begin{aligned}
H_{k|l|m|n|s} &= \alpha_{lmns}H_k + (n-1)\beta_{lmns}y_k - \chi_{lmns}(W_k^i y_i) - v_{lmn}(W_{k|s}^i y_i) \\
&- \frac{1}{4}\omega_{lms}(W_{k|n}^i y_i) - \frac{1}{4}\rho_{lns}(W_{k|m}^i y_i) - \frac{1}{4}c_{lm}(W_{k|n|s}^i y_i) - \frac{1}{4}\eta_{ln}(W_{k|m|s}^i y_i) \\
&- \frac{1}{4}\tau_{ls}(W_{k|m|n}^i y_i) - \frac{1}{4}\gamma_l(W_{k|m|n|s}^i y_i). \quad (3.13)
\end{aligned}$$

If and only if

$$(H_k - y_k R)_{|l|m|n|s} = \alpha_{lmns}(H_k - y_k R). \quad (3.14)$$

And

$$\begin{aligned}
H_{|l|m|n|s} &= \alpha_{lmns}H + (n-1)\beta_{lmns}F^2 \\
&- \frac{1}{3}(3H - F^2 R)_{|l|m|n|s} + \frac{1}{3}\alpha_{lmns}(3H - F^2 R). \quad (4.15)
\end{aligned}$$

This demonstrates that

$$H_{|l|m|n|s} = \alpha_{lmns}H + (n-1)\beta_{lmns}F^2. \quad (3.16)$$

If and only if

$$(3H - F^2 R)_{|l|m|n|s} = \frac{1}{3}\alpha_{lmns}(3H - F^2 R). \quad (3.17)$$

Therefore, the proof of theorem is completed, we can say

Theorem 3.4. In the context of $G^{2nd}R_{|h} - FRF_n$, the h -covariant derivative of fourth-orders for the vector H_k and scalar H are defined in equations (3.13) and (3.16) in [8], respectively, provided that the conditions (3.14) and (3.17) in [8] are satisfied.

By contracting the indices i and h in equation (4.3) and utilizing equations (1.1d), (1.1b), (1.10i), (1.10e), (1.9d) and (1.9b), we obtain the following result:

$$\begin{aligned}
R_{jk|l|m|n|s} &= \alpha_{lmns}R_{jk} + (n-1)\beta_{lmns}g_{jk} + \chi_{lmns}W_{jk} - v_{lmn}W_{jk|s} - \frac{1}{4}\omega_{lms}W_{jk|n} \\
&- \frac{1}{4}c_{lm}W_{jk|n|s} + \frac{1}{4}\rho_{lns}W_{jk|m} - \frac{1}{4}\eta_{ln}W_{jk|m|s} + \frac{1}{4}\tau_{ls}W_{jk|m|n} - \frac{1}{4}\gamma_l W_{jk|m|n|s}
\end{aligned}$$

$$+\frac{1}{3}\alpha_{lmns}(R_{jk}-g_{jk}R)-\frac{1}{3}(R_{jk}-g_{jk}R)_{|l|m|n|s} \quad (3.18)$$

This demonstrates that

$$\begin{aligned} R_{jk|l|m|n|s} &= \alpha_{lmns}R_{jk} + (n-1)\beta_{lmns}g_{jk} + \chi_{lmns}W_{jk} - v_{lmn}W_{jk|s} - \frac{1}{4}\omega_{lms}W_{jk|n} \\ &- \frac{1}{4}c_{lm}W_{jk|n|s} + \frac{1}{4}\rho_{lms}W_{jk|m} - \frac{1}{4}\eta_{ln}W_{jk|m|s} + \frac{1}{4}\tau_{ls}W_{jk|m|n} - \frac{1}{4}\gamma_l W_{jk|m|n|s} \quad (3.19) \end{aligned}$$

If and only if

$$(R_{jk}-g_{jk}R)_{|l|m|n|s} = \alpha_{lmns}(R_{jk}-g_{jk}R) \quad (3.20)$$

In conclusion the proof of theorem is completed, we can say

Theorem 3.5. In the context of $G^{2nd}R|_h - FRF_n$, the h -covariant derivative of fourth-orders for R-Ricci tensor R_{jk} is defined in equation (4.19), provided that the condition (3.20) is satisfied.

By transvecting condition (3.18) with y^k , and applying relations (1.1a), (1.1c), (1.4b), (1.9e) and (1.10d), we obtain the following result

$$R_{j|l|m|n|s} = \alpha_{lmns}R_j + (n-1)\beta_{lmns}y_j + \frac{1}{3}\alpha_{lmns}(R_j - y_jR) - \frac{1}{3}(R_j - y_jR)_{|l|m|n|s} \quad (3.21)$$

This demonstrates that

$$R_{j|l|m|n|s} = \alpha_{lmns}R_j + (n-1)\beta_{lmns}y_j \quad (3.22)$$

If and only if

$$(R_j - y_jR)_{|l|m|n|s} = \alpha_{lmns}(R_j - y_jR) \quad (3.23)$$

By transvecting conditions (3.3) and (3.18) with g^{jk} , respectively, and applying relations (1.2a), (1.2b), (1.9d), (1.10g), and (1.10h), we obtain:

$$\begin{aligned} R_h^i{}_{|l|m|n|s} &= \alpha_{lmns}R_h^i + (n-1)\beta_{lmns}\delta_h^i + (n-1)\chi_{lmns}W_h^i + (n-1)v_{lmn}W_h^i{}_{|s} \\ &+ \frac{1}{4}(n-1)\omega_{lms}W_h^i{}_{|n} + \frac{1}{4}(n-1)c_{lm}W_h^i{}_{|n|s} + \frac{1}{4}(n-1)\rho_{lms}W_h^i{}_{|m} + \frac{1}{4}(n-1)\eta_{ln}W_h^i{}_{|m|s} \\ &+ \frac{1}{4}(n-1)\tau_{ls}W_h^i{}_{|m|n} + \frac{1}{4}(n-1)\gamma_l W_h^i{}_{|m|n|s} + \frac{1}{3}(1-n)\alpha_{lmns}R_h^i - \frac{1}{3}(1-n)R_h^i{}_{|l|m|n|s} \quad (3.24) \end{aligned}$$

This demonstrates that

$$\begin{aligned} R_h^i{}_{|l|m|n|s} &= \alpha_{lmns}R_h^i + (n-1)\beta_{lmns}\delta_h^i + (n-1)\chi_{lmns}W_h^i + (n-1)v_{lmn}W_h^i{}_{|s} \\ &+ \frac{1}{4}\omega_{lms}(n-1)W_h^i{}_{|n} + \frac{1}{4}c_{lm}(n-1)W_h^i{}_{|n|s} + \frac{1}{4}\rho_{lms}(n-1)W_h^i{}_{|m} + \frac{1}{4}(n-1)\eta_{ln}W_h^i{}_{|m|s} \\ &+ \frac{1}{4}(n-1)\tau_{ls}W_h^i{}_{|m|n} + \frac{1}{4}(n-1)\gamma_l W_h^i{}_{|m|n|s} \quad (3.25) \end{aligned}$$

If and only if

$$(R_h^i)_{|l|m|n|s} = \alpha_{lmns}R_h^i \quad (3.26)$$

And

$$\begin{aligned} R_{|l|m|n|s} &= \alpha_{lmns}R + (n-1)\beta_{lmns}n + (n-1)\chi_{lmns}W + (n-1)v_{lmn}W_{|s} \\ &+ \frac{1}{4}(n-1)\omega_{lms}W_{|n} + \frac{1}{4}(n-1)c_{lm}W_{|n|s} + \frac{1}{4}(n-1)\rho_{lms}W_{|m} + \frac{1}{4}(n-1)\eta_{ln}W_{|m|s} \\ &+ \frac{1}{4}(n-1)\tau_{ls}W_{|m|n} + \frac{1}{4}(n-1)\gamma_l W_{|m|n|s} + \frac{1}{3}(1-n)\alpha_{lmns}R - \frac{1}{3}((1-n)R)_{|l|m|n|s} \quad (3.27) \end{aligned}$$

This demonstrates that

$$\begin{aligned} R_{|l|m|n|s} &= \alpha_{lmns}R + (n-1)\beta_{lmns}n + (n-1)\chi_{lmns}W + (n-1)v_{lmn}W_{|s} \\ &+ \frac{1}{4}(n-1)\omega_{lms}W_{|n} + \frac{1}{4}(n-1)c_{lm}W_{|n|s} + \frac{1}{4}(n-1)\rho_{lms}W_{|m} + \frac{1}{4}(n-1)\eta_{ln}W_{|m|s} \end{aligned}$$

$$+ \frac{1}{4}(n-1)\tau_{ls}W_{|m|n} + \frac{1}{4}(n-1)\gamma_l W_{|m|n|s} . \quad (3.28)$$

If and only if

$$(R)_{|l|m|n|s} = \alpha_{lmns} R . \quad (3.29)$$

In conclusion the proof of theorem is completed, we can say

Theorem 3.6. In the context of $G^{2nd}R_{|h} - FRF_n$, the h -covariant derivative of fourth-orders for vector R_j , the projective deviation tensor R_h^i and scalar R are defined in equations (3.22), (3.25) and (3.28), respectively, provided that the conditions (3.23), (3.26) and (3.29), are satisfied.

By transvecting condition (3.3) by g_{ir} , and applying relations (1.1d), (1.2c), (1.9c) and (1.10f), we obtain

$$\begin{aligned} R_{rjkh|l|m|n|s} &= \alpha_{lmns}R_{rjkh} + \beta_{lmns}(g_{rh}g_{jk} - g_{rk}g_{jh}) + \chi_{lmns}(W_{rh}g_{jk} - W_{rk}g_{jh}) \\ &+ v_{lmn}(W_{rh|s}g_{jk} - W_{rk|s}g_{jh}) + \frac{1}{4}\omega_{lms}(W_{rh|n}g_{jk} - W_{rk|n}g_{jh}) \\ &+ \frac{1}{4}c_{lm}(W_{rh|n|s}g_{jk} - W_{rk|n|s}g_{jh}) + \frac{1}{4}\rho_{lns}(W_{rh|m}g_{jk} - W_{rk|m}g_{jh}) \\ &+ \frac{1}{4}\eta_{ln}(W_{rh|m|s}g_{jk} - W_{rk|m|s}g_{jh}) + \frac{1}{4}\tau_{ls}(W_{rh|m|n}g_{jk} - W_{rk|m|n}g_{jh}) \\ &+ \frac{1}{4}\gamma_l(W_{rh|m|n|s}g_{jk} - W_{rk|m|n|s}g_{jh}) + \frac{1}{3}\alpha_{lmns}(g_{rk}R_{jh} - g_{jk}R_{rh}) \\ &- \frac{1}{3}(g_{rk}R_{jh} - g_{jk}R_{rh})_{|l|m|n|s} . \end{aligned} \quad (3.30)$$

This demonstrates that

$$\begin{aligned} R_{rjkh|l|m|n|s} &= \alpha_{lmns}R_{rjkh} + \beta_{lmns}(g_{rh}g_{jk} - g_{rk}g_{jh}) + \chi_{lmns}(W_{rh}g_{jk} - W_{rk}g_{jh}) \\ &+ v_{lmn}(W_{rh|s}g_{jk} - W_{rk|s}g_{jh}) + \frac{1}{4}\omega_{lms}(W_{rh|n}g_{jk} - W_{rk|n}g_{jh}) \\ &+ \frac{1}{4}c_{lm}(W_{rh|n|s}g_{jk} - W_{rk|n|s}g_{jh}) + \frac{1}{4}\rho_{lns}(W_{rh|m}g_{jk} - W_{rk|m}g_{jh}) \\ &+ \frac{1}{4}\eta_{ln}(W_{rh|m|s}g_{jk} - W_{rk|m|s}g_{jh}) + \frac{1}{4}\tau_{ls}(W_{rh|m|n}g_{jk} - W_{rk|m|n}g_{jh}) \\ &+ \frac{1}{4}\gamma_l(W_{rh|m|n|s}g_{jk} - W_{rk|m|n|s}g_{jh}) + \frac{1}{3}\alpha_{lmns}(g_{rk}R_{jh} - g_{jk}R_{rh}) \\ &- \frac{1}{3}(g_{rk}R_{jh} - g_{jk}R_{rh})_{|l|m|n|s} . \end{aligned} \quad (3.31)$$

If and only if

$$(g_{rk}R_{jh} - g_{jk}R_{rh})_{|l|m|n|s} = \alpha_{lmns}(g_{rk}R_{jh} - g_{jk}R_{rh}) . \quad (3.32)$$

Therefore, the proof of theorem is completed, we can say

Theorem 3.7. In the context of $G^{2nd}R_{|h} - FRF_n$, the h -covariant derivative of fourth-orders for associate tensor R_{rjkh} (Cartan's 3th curvature tensor R_{jkh}^i) represents a generalized four-recurrent Finsler space, if and only if the condition (3.32), is satisfied.

It is known that Cartan's 3th curvature tensor R_{jkh}^i and Cartan's 4th curvature tensor K_{jkh}^i are connected by the formula

$$R_{jkh}^i = K_{jkh}^i + C_{jp}^i H_{kh}^p . \quad (3.33)$$

By taking the h -covariant derivative of (3.33), with respect to x^l , x^m , x^n and x^s , respectively, we obtain:

$$R_{jkh|l|m|n|s}^i = k_{jkh|l|m|n|s}^i + (C_{jp}^i H_{kh}^p)_{|l|m|n|s} . \quad (3.34)$$

By substituting equations (3.3) and (3.33) in to (3.34), we obtain:

$$K_{jkh|l|m|n|s}^i = \alpha_{lmns}K_{jkh}^i + \beta_{lmns}(\delta_h^i g_{jk} - \delta_k^i g_{jh}) + \chi_{lmns}(W_h^i g_{jk} - W_k^i g_{jh})$$

$$\begin{aligned}
& + v_{lmn}(W_{h|s}^i g_{jk} - W_{k|s}^i g_{jh}) + \frac{1}{4} \omega_{lms}(W_{h|n}^i g_{jk} - W_{k|n}^i g_{jh}) + \alpha_{lmns}(C_{jp}^i H_{kh}^p) \\
& + \frac{1}{4} c_{lm}(W_{h|n|s}^i g_{jk} - W_{k|n|s}^i g_{jh}) + \frac{1}{4} \rho_{lns}(W_{h|m}^i g_{jk} - W_{k|m}^i g_{jh}) - (C_{jp}^i H_{kh}^p)_{|l|m|n|s} \\
& + \frac{1}{4} \eta_{ln}(W_{h|m|s}^i g_{jk} - W_{k|m|s}^i g_{jh}) + \frac{1}{4} \tau_{ls}(W_{h|m|n}^i g_{jk} - W_{k|m|n}^i g_{jh}) \\
& + \frac{1}{4} \gamma_l(W_{h|m|n|s}^i g_{jk} - W_{k|m|n|s}^i g_{jh}) + \frac{1}{3} \alpha_{lmns}(\delta_k^i R_{jh} - g_{jk} R_h^i) \\
& - \frac{1}{3}(\delta_k^i R_{jh} - g_{jk} R_h^i)_{|l|m|n|s}.
\end{aligned} \tag{3.35}$$

This demonstrates that

$$\begin{aligned}
K_{jkh|l|m|n|s}^i & = \alpha_{lmns} K_{jkh}^i + \beta_{lmns}(\delta_h^i g_{jk} - \delta_k^i g_{jh}) + \chi_{lmns}(W_{h|s}^i g_{jk} - W_{k|s}^i g_{jh}) \\
& + v_{lmn}(W_{h|s}^i g_{jk} - W_{k|s}^i g_{jh}) + \frac{1}{4} \omega_{lms}(W_{h|n}^i g_{jk} - W_{k|n}^i g_{jh}) \\
& + \frac{1}{4} c_{lm}(W_{h|n|s}^i g_{jk} - W_{k|n|s}^i g_{jh}) + \frac{1}{4} \rho_{lns}(W_{h|m}^i g_{jk} - W_{k|m}^i g_{jh}) \\
& + \frac{1}{4} \eta_{ln}(W_{h|m|s}^i g_{jk} - W_{k|m|s}^i g_{jh}) + \frac{1}{4} \tau_{ls}(W_{h|m|n}^i g_{jk} - W_{k|m|n}^i g_{jh}) \\
& + \frac{1}{4} \gamma_l(W_{h|m|n|s}^i g_{jk} - W_{k|m|n|s}^i g_{jh}) .
\end{aligned} \tag{3.36}$$

If and only if

$$(\delta_k^i R_{jh} - g_{jk} R_h^i)_{|l|m|n|s} = \alpha_{lmns}(\delta_k^i R_{jh} - g_{jk} R_h^i). \tag{3.37}$$

and

$$(C_{jp}^i H_{kh}^p)_{|l|m|n|s} = \alpha_{lmns}(C_{jp}^i H_{kh}^p). \tag{3.38}$$

Therefore, the proof of theorem is completed, we can say

Theorem 3.8. In the context of $G^{2nd}R_{|h} - FFR_n$, Cartan's 4th curvature tensor K_{jkh}^i defines a generalized four-recurrent Finsler space if and only if the tensors $(\delta_k^i R_{jh} - g_{jk} R_h^i)$ and $(C_{jp}^i H_{kh}^p)$ are a generalized four-recurrent Finsler space.

By contracting the indices i and h in equation (3.35) and utilizing equations (1.1d), (1.1b), (1.10e), (1.11e), (1.9c) and (1.9b), we obtain the following result:

$$\begin{aligned}
K_{jk|l|m|n|s} & = \alpha_{lmns} K_{jk} + \beta_{lmns}(n-1)g_{jk} + \chi_{lmns} W_{jk} - v_{lmn} W_{jk|s} - \frac{1}{4} \omega_{lms} W_{jk|n} \\
& - \frac{1}{4} c_{lm} W_{jk|n|s} + \frac{1}{4} \rho_{lns} W_{jk|m} - \frac{1}{4} \eta_{ln} W_{jk|m|s} + \frac{1}{4} \tau_{ls} W_{jk|m|n} - \frac{1}{4} \gamma_l W_{jk|m|n|s} \\
& + \frac{1}{3} \alpha_{lmns}(R_{jk} - g_{jk} R) - \frac{1}{3}(R_{jk} - g_{jk} R)_{|l|m|n|s} + \alpha_{lmns}(C_{jp}^i H_{ki}^p) - (C_{jp}^i H_{ki}^p)_{|l|m|n|s}.
\end{aligned} \tag{3.39}$$

This demonstrates that

$$\begin{aligned}
K_{jk|l|m|n|s} & = \alpha_{lmns} K_{jk} + \beta_{lmns}(n-1)g_{jk} + \chi_{lmns} W_{jk} - v_{lmn} W_{jk|s} - \frac{1}{4} \omega_{lms} W_{jk|n} \\
& - \frac{1}{4} c_{lm} W_{jk|n|s} + \frac{1}{4} \rho_{lns} W_{jk|m} - \frac{1}{4} \eta_{ln} W_{jk|m|s} + \frac{1}{4} \tau_{ls} W_{jk|m|n} - \frac{1}{4} \gamma_l W_{jk|m|n|s} .
\end{aligned} \tag{3.40}$$

If and only if

$$(R_{jk} - g_{jk} R)_{|l|m|n|s} = \alpha_{lmns}(R_{jk} - g_{jk} R). \tag{3.41}$$

and

$$(C_{jp}^i H_{ki}^p)_{|l|m|n|s} = \alpha_{lmns}(C_{jp}^i H_{ki}^p). \tag{4.42}$$

By transvecting condition (3.39) with y^k , and applying relations (1.1a), (1.1c), (1.4b), (1.9a), (1.9e), (1.11c) and (1.10d), we obtain the following result

$$K_{j|l|m|n|s} = \alpha_{lmns} K_j + \beta_{lmns}(n-1)y_j + \frac{1}{3} \alpha_{lmns}(R_j - y_j R) - \frac{1}{3}(R_j - y_j R)_{|l|m|n|s}$$

$$+ \alpha_{lmns}(C_{jp}^i H_i^p) - (C_{jp}^i H_i^p)_{|l|m|n|s} . \quad (3.43)$$

This demonstrates that

$$K_{j|l|m|n|s} = \alpha_{lmns} K_j + (n-1)\beta_{lmns} \mathcal{Y}_j . \quad (3.44)$$

If and only if

$$(R_j - y_j R)_{|l|m|n|s} = \alpha_{lmns} (R_j - y_j R) . \quad (4.45)$$

and

$$(C_{jp}^i H_i^p)_{|l|m|n|s} = \alpha_{lmns} (C_{jp}^i H_i^p) . \quad (3.46)$$

Therefore, the proof of theorem is completed, we conclude

Theorem 3.9. In the context of $G^{2nd}R_{|h} - FRR_n$, the h -covariant derivative of fourth-orders for the K-Ricci tensor K_{jk} and curvature vector K_j are defined in equations (3.40) and (3.44), respectively, provided that the conditions (3.41), (3.42), (3.45) and (3.46) are satisfied.

By transvecting conditions (3.35) and (3.39) with g^{jk} , respectively, and applying relations (1.2a), (1.2b), (1.9d), (1.11d) and (1.11f), we obtain:

$$\begin{aligned} K_{h|l|m|n|s}^i &= \alpha_{lmns} K_h^i + (n-1)\beta_{lmns} \delta_h^i + (n-1)\chi_{lmns} W_h^i + (n-1)v_{lmn} W_{h|s}^i \\ &+ \frac{1}{4}(n-1)\omega_{lms} W_{h|n}^i + \frac{1}{4}(n-1)c_{lm} W_{h|n|s}^i + \frac{1}{4}(n-1)\rho_{lms} W_{h|m}^i + \frac{1}{4}(n-1)\eta_{ln} W_{h|m|s}^i \\ &+ \frac{1}{4}(n-1)\tau_{ls} W_{h|m|n}^i + \frac{1}{4}(n-1)\gamma_l W_{h|m|n|s}^i + \frac{1}{3}(1-n)\alpha_{lmns} R_h^i + \alpha_{lmns} g^{jk} (C_{jp}^i H_{kh}^p) \\ &- \frac{1}{3}(1-n)R_{h|l|m|n|s}^i - g^{jk} (C_{jp}^i H_{kh}^p)_{|l|m|n|s} . \end{aligned} \quad (3.47)$$

This demonstrates that

$$\begin{aligned} K_{h|l|m|n|s}^i &= \alpha_{lmns} K_h^i + (n-1)\beta_{lmns} \delta_h^i + (n-1)\chi_{lmns} W_h^i + (n-1)v_{lmn} W_{h|s}^i \\ &+ \frac{1}{4}(n-1)\omega_{lms} W_{h|n}^i + \frac{1}{4}(n-1)c_{lm} W_{h|n|s}^i + \frac{1}{4}(n-1)\rho_{lms} W_{h|m}^i + \frac{1}{4}(n-1)\eta_{ln} W_{h|m|s}^i \\ &+ \frac{1}{4}(n-1)\tau_{ls} W_{h|m|n}^i + \frac{1}{4}(n-1)\gamma_l W_{h|m|n|s}^i . \end{aligned} \quad (3.48)$$

If and only if

$$(C_{jp}^i H_{kh}^p)_{|l|m|n|s} = \alpha_{lmns} (C_{jp}^i H_{kh}^p), \text{ where } g^{jk} \neq 0 . \text{ and} \quad (3.49)$$

$$R_{h|l|m|n|s}^i = \alpha_{lmns} R_h^i . \quad (3.50)$$

And

$$\begin{aligned} K_{|l|m|n|s} &= \alpha_{lmns} K + (n-1)n\beta_{lmns} + (n-1)\chi_{lmns} W + (n-1)v_{lmn} W_{|s} \\ &+ \frac{1}{4}(n-1)\omega_{lms} W_{|n} + \frac{1}{4}(n-1)c_{lm} W_{|n|s} + \frac{1}{4}(n-1)\rho_{lms} W_{|m} + \frac{1}{4}(n-1)\eta_{ln} W_{|m|s} \\ &+ \frac{1}{4}(n-1)\tau_{ls} W_{|m|n} + \frac{1}{4}(n-1)\gamma_l W_{|m|n|s} + \frac{1}{3}(1-n)\alpha_{lmns} R + \alpha_{lmns} g^{jk} (C_{jp}^i H_i^p) \\ &- \frac{1}{3}((1-n)R)_{|l|m|n|s} - g^{jk} (C_{jp}^i H_i^p)_{|l|m|n|s} . \end{aligned} \quad (3.51)$$

This demonstrates that

$$\begin{aligned} K_{|l|m|n|s} &= \alpha_{lmns} K + (n-1)n\beta_{lmns} + (n-1)\chi_{lmns} W + (n-1)v_{lmn} W_{|s} \\ &+ \frac{1}{4}(n-1)\omega_{lms} W_{|n} + \frac{1}{4}(n-1)c_{lm} W_{|n|s} + \frac{1}{4}(n-1)\rho_{lms} W_{|m} + \frac{1}{4}(n-1)\eta_{ln} W_{|m|s} \\ &+ \frac{1}{4}(n-1)\tau_{ls} W_{|m|n} + \frac{1}{4}(n-1)\gamma_l W_{|m|n|s} . \end{aligned} \quad (3.52)$$

If and only if

$$(C_{jp}^i H_{kh}^p)_{|l|m|n|s} = \alpha_{lmns} (C_{jp}^i H_{kh}^p) \quad , \quad (3.53)$$

where $g^{jk} \neq 0$, and

$$R_{|l|m|n|s} = \alpha_{lmns} R. \quad (3.54)$$

Therefore, the proof of theorem is completed, we conclude

Theorem 3.10. In the context of $G^{2nd}R_{|h} - FRF_n$, the h -covariant derivative of fourth-orders for the projective deviation tensor K_h^i and scalar K are defined in equations (3.48) and (3.52), respectively, provided that the conditions (3.49), (3.50), (3.53) and (3.54) are satisfied.

By transvecting condition (3.35) by g_{ir} , and applying relations (1.1d), (1.2c), (1.3c), (1.9c), (1.10f) and (1.11a), we obtain

$$\begin{aligned} K_{rjkh|l|m|n|s} &= \alpha_{lmns} K_{rjkh} + \beta_{lmns} (g_{rh} g_{jk} - g_{rk} g_{jh}) + \chi_{lmns} (W_{rh} g_{jk} - W_{rk} g_{jh}) \\ &+ v_{lmn} (W_{rh|s} g_{jk} - W_{rk|s} g_{jh}) + \frac{1}{4} \omega_{lms} (W_{rh|n} g_{jk} - W_{rk|n} g_{jh}) \\ &+ \frac{1}{4} c_{lm} (W_{rh|n|s} g_{jk} - W_{rk|n|s} g_{jh}) + \frac{1}{4} \rho_{lns} (W_{rh|m} g_{jk} - W_{rk|m} g_{jh}) \\ &+ \frac{1}{4} \eta_{ln} (W_{rh|m|s} g_{jk} - W_{rk|m|s} g_{jh}) + \frac{1}{4} \tau_{ls} (W_{rh|m|n} g_{jk} - W_{rk|m|n} g_{jh}) \\ &+ \frac{1}{4} \gamma_l (W_{rh|m|n|s} g_{jk} - W_{rk|m|n|s} g_{jh}) + \frac{1}{3} \alpha_{lmns} (g_{rk} R_{jh} - g_{jk} R_{rh}) \\ &- \frac{1}{3} (g_{rk} R_{jh} - g_{jk} R_{rh})_{|l|m|n|s} - (C_{rjp} H_{kh}^p)_{|l|m|n|s} + \alpha_{lmns} (C_{rjp} H_{kh}^p). \end{aligned} \quad (3.55)$$

This demonstrates that

$$\begin{aligned} K_{rjkh|l|m|n|s} &= \alpha_{lmns} K_{rjkh} + \beta_{lmns} (g_{rh} g_{jk} - g_{rk} g_{jh}) + \chi_{lmns} (W_{rh} g_{jk} - W_{rk} g_{jh}) \\ &+ v_{lmn} (W_{rh|s} g_{jk} - W_{rk|s} g_{jh}) + \frac{1}{4} \omega_{lms} (W_{rh|n} g_{jk} - W_{rk|n} g_{jh}) \\ &+ \frac{1}{4} c_{lm} (W_{rh|n|s} g_{jk} - W_{rk|n|s} g_{jh}) + \frac{1}{4} \rho_{lns} (W_{rh|m} g_{jk} - W_{rk|m} g_{jh}) \\ &+ \frac{1}{4} \eta_{ln} (W_{rh|m|s} g_{jk} - W_{rk|m|s} g_{jh}) + \frac{1}{4} \tau_{ls} (W_{rh|m|n} g_{jk} - W_{rk|m|n} g_{jh}) \\ &+ \frac{1}{4} \gamma_l (W_{rh|m|n|s} g_{jk} - W_{rk|m|n|s} g_{jh}). \end{aligned} \quad (3.56)$$

If and only if

$$(g_{rk} R_{jh} - g_{jk} R_{rh})_{|l|m|n|s} = \alpha_{lmns} (g_{rk} R_{jh} - g_{jk} R_{rh}). \quad (3.57)$$

And

$$(C_{rjp} H_{kh}^p)_{|l|m|n|s} = \alpha_{lmns} (C_{rjp} H_{kh}^p). \quad (3.58)$$

Therefore, the proof of theorem is completed, we can say

Theorem 3.11. In the context of $G^{2nd}R_{|h} - FRF_n$, the h -covariant derivative of fourth-orders for the associate tensor K_{rjkh} (Cartan's 4th curvature tensor K_{jkh}^i) represents a generalized four-recurrent Finsler space, if and only if the condition (3.57) and (3.58) are satisfied.

It is known that Cartan's 4th curvature tensor K_{jkh}^i and Berwald curvature tensor H_{jkh}^i are connected by the formula

$$H_{jkh}^i = K_{jkh}^i + y^r (\partial_j K_{rkh}^i). \quad (3.59)$$

By taking the h -covariant derivative of (3.59), with respect to x^l , x^m , x^n and x^s , using (1.4b), we obtain:

$$H_{jkh|l|m|n|s}^i = K_{jkh|l|m|n|s}^i + y^r (\partial_j K_{rkh}^i)_{|l|m|n|s}. \quad (3.60)$$

By substituting equations (3.35) and (3.59) in to (3.60), we obtain:

$$\begin{aligned}
H_{jkh|l|m|n|s}^i &= \alpha_{lmns}H_{jkh}^i + \beta_{lmns}(\delta_h^i g_{jk} - \delta_k^i g_{jh}) + \chi_{lmns}(W_h^i g_{jk} - W_k^i g_{jh}) \\
&+ v_{lmn}(W_{h|s}^i g_{jk} - W_{k|s}^i g_{jh}) + \frac{1}{4}\omega_{lms}(W_{h|n}^i g_{jk} - W_{k|n}^i g_{jh}) + \alpha_{lmns}(C_{jp}^i H_{kh}^p) \\
&+ \frac{1}{4}c_{lm}(W_{h|n|s}^i g_{jk} - W_{k|n|s}^i g_{jh}) + \frac{1}{4}\rho_{lns}(W_{h|m}^i g_{jk} - W_{k|m}^i g_{jh}) - (C_{jp}^i H_{kh}^p)_{|l|m|n|s} \\
&+ \frac{1}{4}\eta_{ln}(W_{h|m|s}^i g_{jk} - W_{k|m|s}^i g_{jh}) + \frac{1}{4}\tau_{ls}(W_{h|m|n}^i g_{jk} - W_{k|m|n}^i g_{jh}) \\
&+ \frac{1}{4}\gamma_l(W_{h|m|n|s}^i g_{jk} - W_{k|m|n|s}^i g_{jh}) + \frac{1}{3}\alpha_{lmns}(\delta_k^i R_{jh} - g_{jk} R_h^i) - \alpha_{lmns}y^r(\partial_j K_{rkh}^i) \\
&- \frac{1}{3}(\delta_k^i R_{jh} - g_{jk} R_h^i)_{|l|m|n|s} + y^r(\partial_j K_{rkh}^i)_{|l|m|n|s} .
\end{aligned} \tag{3.61}$$

This demonstrates that

$$\begin{aligned}
H_{jkh|l|m|n|s}^i &= \alpha_{lmns}H_{jkh}^i + \beta_{lmns}(\delta_h^i g_{jk} - \delta_k^i g_{jh}) + \chi_{lmns}(W_h^i g_{jk} - W_k^i g_{jh}) \\
&+ v_{lmn}(W_{h|s}^i g_{jk} - W_{k|s}^i g_{jh}) + \frac{1}{4}\omega_{lms}(W_{h|n}^i g_{jk} - W_{k|n}^i g_{jh}) + \alpha_{lmns}(C_{jp}^i H_{kh}^p) \\
&+ \frac{1}{4}c_{lm}(W_{h|n|s}^i g_{jk} - W_{k|n|s}^i g_{jh}) + \frac{1}{4}\rho_{lns}(W_{h|m}^i g_{jk} - W_{k|m}^i g_{jh}) - (C_{jp}^i H_{kh}^p)_{|l|m|n|s} \\
&+ \frac{1}{4}\eta_{ln}(W_{h|m|s}^i g_{jk} - W_{k|m|s}^i g_{jh}) + \frac{1}{4}\tau_{ls}(W_{h|m|n}^i g_{jk} - W_{k|m|n}^i g_{jh}) \\
&+ \frac{1}{4}\gamma_l(W_{h|m|n|s}^i g_{jk} - W_{k|m|n|s}^i g_{jh}) .
\end{aligned} \tag{3.62}$$

If and only if

$$(\delta_k^i R_{jh} - g_{jk} R_h^i)_{|l|m|n|s} = \alpha_{lmns}(\delta_k^i R_{jh} - g_{jk} R_h^i) , \tag{3.63}$$

$$(C_{jp}^i H_{kh}^p)_{|l|m|n|s} = \alpha_{lmns}(C_{jp}^i H_{kh}^p) . \tag{3.64}$$

and

$$(\partial_j K_{rkh}^i)_{|l|m|n|s} = \alpha_{lmns}(\partial_j K_{rkh}^i) , \text{ where } y^r \neq 0 . \tag{3.65}$$

Therefore, the proof of theorem is completed, we can say

Theorem 3.12. In the context of $G^{2nd}R_{|h} - FRF_n$, Berwald curvature tensor H_{jkh}^i defines a generalized four-recurrent Finsler space if and only if the tensors $(\delta_k^i R_{jh} - g_{jk} R_h^i)$, $(C_{jp}^i H_{kh}^p)$ and $(\partial_j K_{rkh}^i)$ are a generalized four-recurrent Finsler space.

By contracting the indices i and h in equation (3.61) and utilizing equations (1.1d), (1.1b), (1.2a), (1.10e), (1.10m), (1.11e), (1.9c) and (1.9b), we obtain the following result:

$$\begin{aligned}
H_{jk|l|m|n|s} &= \alpha_{lmns}H_{jk} + \beta_{lmns}(n-1)g_{jk} + \chi_{lmns}W_{jk} + v_{lmn}W_{jk|s} + \frac{1}{4}\omega_{lms}W_{jk|n} \\
&+ \frac{1}{4}c_{lm}W_{jk|n|s} + \frac{1}{4}\rho_{lns}W_{jk|m} + \frac{1}{4}\eta_{ln}W_{jk|m|s} + \frac{1}{4}\tau_{ls}W_{jk|m|n} + \frac{1}{4}\gamma_l W_{jk|m|n|s} \\
&- (C_{jp}^i H_{ki}^p)_{|l|m|n|s} + \alpha_{lmns}(C_{jp}^i H_{ki}^p) + \frac{1}{3}\alpha_{lmns}(R_{jk} - g_{jk}R) - \alpha_{lmns}y^r(\partial_j K_{rk}) \\
&- \frac{1}{3}(R_{jk} - g_{jk}R)_{|l|m|n|s} + y^r(\partial_j K_{rk})_{|l|m|n|s} .
\end{aligned} \tag{3.66}$$

This demonstrates that

$$\begin{aligned}
H_{jk|l|m|n|s} &= \alpha_{lmns}H_{jk} + \beta_{lmns}(n-1)g_{jk} + \chi_{lmns}W_{jk} + v_{lmn}W_{jk|s} + \frac{1}{4}\omega_{lms}W_{jk|n} \\
&+ \frac{1}{4}c_{lm}W_{jk|n|s} + \frac{1}{4}\rho_{lns}W_{jk|m} + \frac{1}{4}\eta_{ln}W_{jk|m|s} + \frac{1}{4}\tau_{ls}W_{jk|m|n} + \frac{1}{4}\gamma_l W_{jk|m|n|s} .
\end{aligned} \tag{3.67}$$

If and only if

$$(R_{jk} - g_{jk}R)_{|l|m|n|s} = \alpha_{lmns}(R_{jk} - g_{jk}R) , \tag{3.68}$$

$$(C_{jp}^i H_{ki}^p)_{|l|m|n|s} = \alpha_{lmns}(C_{jp}^i H_{ki}^p) , \tag{3.69}$$

and

$$(\partial_j K_{rk})_{|l|m|n|s} = \alpha_{lmns}(\partial_j K_{rk}), \text{ where } y^r \neq 0. \quad (3.70)$$

Therefore, the proof of theorem is completed, we can say

Theorem 3.13. In the context of $G^{2nd}R_{|h} - FRF_n$, Berwald H–Ricci tensor H_{jk} defines a generalized four-recurrent Finsler space if and only if the tensors $(R_{jk} - g_{jk} R)$, $(C_{jp}^i H_{ki}^p)$ and $(\partial_j K_{sk})$ are a generalized four-recurrent Finsler space.

By contracting the indices i and j in equation (3.61) and utilizing equations (1.1d), (1.1b), (1.3b), (1.10e), (1.9c) and (1.10l), we obtain the following result:

$$\begin{aligned} (H_{hk} - H_{kh})_{|l|m|n|s} &= \alpha_{lmns}(H_{hk} - H_{kh}) + \chi_{lmns}(W_{hk} - W_{kh}) + \alpha_{lmns}(C_p H_{kh}^p) \\ &+ v_{lmn}(W_{hk|s} - W_{kh|s}) + \frac{1}{4}\omega_{lms}(W_{hk|n} - W_{kh|n}) + \frac{1}{4}c_{lm}(W_{hk|n|s} - W_{kh|n|s}) \\ &+ \frac{1}{4}\rho_{lns}(W_{hk|m} - W_{kh|m}) + \frac{1}{4}\eta_{ln}(W_{hk|m|s} - W_{kh|m|s}) + \frac{1}{4}\tau_{ls}(W_{hk|m|n} - W_{kh|m|n}) \\ &+ \frac{1}{4}\gamma_l(W_{hk|m|n|s} - W_{kh|m|n|s}) - \alpha_{lmns}y^r(\partial_i K_{rkh}^i) + y^r(\partial_i K_{rkh}^i)_{|l|m|n|s} - (C_p H_{kh}^p)_{|l|m|n|s}. \end{aligned} \quad (3.71)$$

This demonstrates that

$$\begin{aligned} (H_{hk} - H_{kh})_{|l|m|n|s} &= \alpha_{lmns}(H_{hk} - H_{kh}) + \chi_{lmns}(W_{hk} - W_{kh}) \\ &+ v_{lmn}(W_{hk|s} - W_{kh|s}) + \frac{1}{4}\omega_{lms}(W_{hk|n} - W_{kh|n}) + \frac{1}{4}c_{lm}(W_{hk|n|s} - W_{kh|n|s}) \\ &+ \frac{1}{4}\rho_{lns}(W_{hk|m} - W_{kh|m}) + \frac{1}{4}\eta_{ln}(W_{hk|m|s} - W_{kh|m|s}) + \frac{1}{4}\tau_{ls}(W_{hk|m|n} - W_{kh|m|n}) \\ &+ \frac{1}{4}\gamma_l(W_{hk|m|n|s} - W_{kh|m|n|s}). \end{aligned} \quad (3.72)$$

If and only if

$$(C_p H_{kh}^p)_{|l|m|n|s} = \alpha_{lmns}(C_p H_{kh}^p), \quad \text{and} \quad (3.73)$$

$$(\partial_i K_{rkh}^i)_{|l|m|n|s} = \alpha_{lmns}(\partial_i K_{rkh}^i), \text{ where } y^r \neq 0. \quad (3.74)$$

Therefore, the proof of theorem is completed, we can say

Theorem 3.14. In the context of $G^{2nd}R_{|h} - FRF_n$, the tensor $(H_{hk} - H_{kh})$ defines a generalized four-recurrent Finsler space if and only if the tensors $(C_p H_{kh}^p)$ and $(\partial_i K_{rkh}^i)$ are a generalized four-recurrent Finsler space.

By taking the h -covariant derivative of (1.11h), with respect to x^l, x^m, x^n and x^s , respectively, and using (1.4b), we obtain:

$$H_{jkh|l|m|n|s}^i - K_{jkh|l|m|n|s}^i = \left(P_{jk|h}^i + P_{jk}^r P_{rh}^i - \frac{h}{k}\right)_{|l|m|n|s}. \quad (3.75)$$

From the equations (3.35) and (3.61), we get

$$\begin{aligned} H_{jkh|l|m|n|s}^i - K_{jkh|l|m|n|s}^i &= \alpha_{lmns}(H_{jkh}^i - K_{jkh}^i) \\ &+ y^r(\partial_j K_{rkh}^i)_{|l|m|n|s} - \alpha_{lmns}y^r(\partial_j K_{rkh}^i). \end{aligned} \quad (3.76)$$

By substituting equations (3.76) and (1.11h) in to (3.75), we obtain:

$$\begin{aligned} \left(P_{jk|h}^i + P_{jk}^r P_{rh}^i - \frac{h}{k}\right)_{|l|m|n|s} &= \alpha_{lmns} \left(P_{jk|h}^i + P_{jk}^r P_{rh}^i - \frac{h}{k}\right) \\ &+ y^r(\partial_j K_{rkh}^i)_{|l|m|n|s} - \alpha_{lmns}y^r(\partial_j K_{rkh}^i). \end{aligned} \quad (3.77)$$

This demonstrates that

$$\left(P_{jk|h}^i + P_{jk}^r P_{rh}^i - \frac{h}{k}\right)_{|l|m|n|s} = \alpha_{lmns} \left(P_{jk|h}^i + P_{jk}^r P_{rh}^i - \frac{h}{k}\right). \quad (3.78)$$

If and only if

$$(\partial_j K_{rkh}^i)_{|l|m|n|s} = \alpha_{lmns} (\partial_j K_{rkh}^i), \text{ where } y^r \neq 0. \quad (3.79)$$

Thus, the proof of theorem is completed, we can say

Theorem 3.15. In $G^{2nd}R_{|h} - FRF_n$, the tensor $\left(P_{jk|h}^i + P_{jk}^r P_{rh}^i - \frac{h}{k}\right)$ is a generalized four-recurrent Finsler space if and only if the tensor $(\partial_j K_{rkh}^i)$ is a generalized four-recurrent Finsler space.

DISCUSSION

Biomedical Interpretation of the Mathematical Findings

The mathematical results obtained in this study indicate that higher-order recurrence imposes additional structural constraints on curvature tensors in generalized Finsler spaces. From a biomedical engineering perspective, these properties may be relevant to systems characterized by directional dependence and spatial anisotropy, including certain biological structures and diffusion-based imaging representations.

The direction-dependent nature of the Finsler metric provides a theoretical framework for representing anisotropic geometric structures. In this context, the curvature and recurrence relations developed in the present study may have potential relevance to the geometric representation of neural fiber organization, vascular architecture, and anisotropic tissue structures. These observations should be interpreted as theoretical relevance rather than as evidence of a validated biomedical application. The present study does not use clinical imaging data or experimentally derived biomedical measurements. Consequently, the practical significance of the proposed framework remains to be established through future computational and experimental investigations.

LIMITATIONS

The principal limitation of this study is its theoretical nature. No patient-derived data, diffusion MRI datasets, histological data, vascular imaging data, or biomechanical measurements were analyzed. Consequently, the proposed mathematical framework has not been experimentally or clinically validated.

A second limitation is that the biomedical applications discussed in this manuscript remain conceptual. Although Finsler geometry provides a mathematical framework for representing direction-dependent structures, the present study does not establish whether the proposed higher-order recurrence conditions improve the performance of any specific biomedical imaging or diagnostic task.

Third, no computational implementation or quantitative comparison with established Riemannian, tensor-based, or other geometric imaging approaches was performed. Therefore, the practical performance,

computational efficiency, stability, and reproducibility of the proposed framework remain to be determined.

Finally, the translation of the theoretical results into biomedical applications will require validation using appropriate real-world datasets and independent experimental or computational studies.

FUTURE RESEARCH DIRECTIONS

Future research should focus on translating the theoretical framework developed in this study into computational applications. An initial step would be to construct Finsler metrics from real diffusion MRI data and investigate whether higher-order curvature information can improve the geometric representation of anisotropic tissue structures.

The proposed framework could subsequently be evaluated in neural fiber tractography, vascular network characterization, biomechanical tissue modeling, and orientation-sensitive medical image analysis. These investigations should include comparisons with conventional Riemannian and tensor-based approaches using objective measures of accuracy, stability, computational efficiency, and reproducibility.

Machine-learning methods may also be investigated to determine whether higher-order Finsler curvature descriptors can serve as geometric features for image classification, segmentation, or other image-analysis tasks. Such studies should use appropriately validated datasets and predefined performance measures.

Ultimately, independent computational and experimental studies using clinically relevant datasets will be required to determine the practical and clinical utility of the proposed framework.

CONCLUSION

This study developed a theoretical framework for higher-order generalized four-recurrent Finsler spaces using higher-order Cartan covariant differentiation. The analysis established relationships involving Weyl's projective curvature tensor, Cartan curvature tensors, Berwald curvature tensor, Ricci tensors, torsion tensors, and deviation tensors, together with corresponding characterization conditions for generalized higher-order recurrence.

The results contribute to the theoretical development of recurrent structures in Finsler geometry and provide a basis for further investigation of higher-order

curvature relationships. Owing to the direction-dependent nature of the Finsler metric, the framework may also have potential relevance to the mathematical representation of anisotropic structures encountered in biomedical imaging and biomedical engineering.

The biomedical implications proposed in this study are theoretical and should not be interpreted as evidence of clinical or diagnostic effectiveness. Future computational and experimental investigations using appropriate biomedical datasets will be necessary to determine whether the proposed geometric framework offers practical advantages in biomedical image analysis and related applications.

DECLARATIONS

Ethics Approval and Consent to Participate

Not applicable. This study is theoretical and mathematical in nature and does not involve human participants, patient-derived data, animal experiments, or clinical interventions.

Consent for Publication

Not applicable. No individual patient data, identifiable information, or personal images are included in this manuscript.

Availability of Data and Materials

No clinical or experimental datasets were generated or analyzed in this theoretical study. The mathematical results and derivations presented in the manuscript are available within the article.

Competing Interests

The authors declare that they have no competing interests.

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Authors' Contributions

All authors contributed to the conception and development of the mathematical framework, analysis and interpretation of the theoretical results, manuscript preparation, critical revision of the manuscript, and approval of the final version.

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